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THE SEMI-PERFECT NUMBER

608. [January, 1966] *Proposed by A. A. Gioia and A. M. Vaidya, Texas Technological College.*

Call a positive integer n semi-perfect if the sum of all the square free divisors of n is $2n$. Prove that 6 is the only semi-perfect number.

Solution by Stanley Rabinowitz, Far Rockaway, New York.

Suppose

$$n = (p_1)^{a_1}(p_2)^{a_2}(p_3)^{a_3} \cdots (p_k)^{a_k}(p_i < p_{i+1}).$$

The sum of the square free divisors of n is the same as the sum of the divisors of $p_1p_2p_3 \cdots p_k$ which is $(p_1+1)(p_2+1) \cdots (p_k+1)$. If n is semi-perfect, then

$$(1) \quad 2(p_1)^{a_1}(p_2)^{a_2}(p_3)^{a_3} \cdots (p_k)^{a_k} = (p_1 + 1)(p_2 + 1)(p_3 + 1) \cdots (p_k + 1).$$

It is clear that $k > 1$. Hence n must be even; for if it were odd, then 2 would divide the left of (1), but 4 would divide the right. Therefore $p_1 = 2$, $p_1 + 1 = 3$, so $p_2 = 3$. Now p_k divides the left of (1) but not the right ($k > 2$). Hence $k = 2$. Therefore we find $a_1 = 1$ and $a_2 = 1$. Therefore 6 is the only semi-perfect number.